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Banach Function Spaces and Interpolation Methods III. Hausdorff-Young Estimates

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1. INTRODUCTION

When $\mathbf{R}$ denotes the real line, the Fourier transform $\mathscr{F}$ is defined on $L^1(\mathbf{R})$ by

$$(\mathscr{F}f)(y) = \int_{-\infty}^{\infty} f(x)\, e^{-ixy}\, dx, \qquad -\infty < y < \infty.$$

A simple consequence of the definition is that $\mathscr{F}$ is a bounded linear operator from $L^1(\mathbf{R})$ into $L^\infty(\mathbf{R})$. Lying a little deeper is the theorem of Plancherel [8, p. 26] which asserts that $\mathscr{F}$ has a linear extension to all of $L^2(\mathbf{R})$ which maps $L^2(\mathbf{R})$ isometrically onto itself. To complete the picture, at least for Lebesgue spaces, there is the theorem of Hausdorff and Young [9, p. 101] which describes the behavior of $\mathscr{F}$ on the intermediate spaces $L^p(\mathbf{R})$, $1 < p < 2$.

THEOREM 1.1 (Hausdorff–Young). *Suppose $1 < p < 2$ and let p' be the conjugate exponent to p, i.e., $p^{-1} + (p')^{-1} = 1$. Then $\mathscr{F}$ is a bounded linear operator from $L^p(\mathbf{R})$ into $L^{p'}(\mathbf{R})$.*

When $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, the Lorentz space $L^{pq}(\mathbf{R})$ consists of those measurable functions f on $\mathbf{R}$ for which the norm

$$\|f\|_{pq} = \left\{\int_0^\infty [t^{1/p} f^{**}(t)]^q\, dt/t\right\}^{1/q}$$

is finite. Here, f^{**} denotes the integral average $f^{**}(t) = t^{-1}\int_0^t f^*(s)\, ds$ of the decreasing rearrangement f^* of f. The next theorem is essentially due to Hardy and Littlewood [9, p. 128].

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THEOREM 1.2 (Hardy–Littlewood). *If* $1 < p < 2$ *and* p' *is conjugate to* p, *then* $\mathscr{F}$ *is a bounded operator from* $L^p(\mathbf{R})$ *into* $L^{p'p}(\mathbf{R})$.

Since $p < p'$ we have $L^{p'p}(\mathbf{R}) \subseteq L^{p'p'}(\mathbf{R}) = L^{p'}(\mathbf{R})$ so Theorem 1.2 constitutes a sharpening of Theorem 1.1. More generally, the action of $\mathscr{F}$ on any Lorentz space L^{pq} is described by the interpolation theorem of Calderón [5].

THEOREM 1.3 (Calderón). *If* $1 < p < 2$, $1 \leqslant q \leqslant \infty$, *and* p' *is conjugate to* p, *then* $\mathscr{F}$ *is a bounded linear operator from* $L^{pq}(\mathbf{R})$ *into* $L^{p'q}(\mathbf{R})$.

We shall prove the following Hausdorff–Young-type theorem for rearrangement–invariant (r.i.) spaces.

THEOREM 1.4. *Let* $L^\mu(\mathbf{R})$ *be a* r.i. *space whose indices* (β, α) *satisfy* $\frac{1}{2} < \beta \leqslant \alpha < 1$. *Then the functional* $\hat{\mu}$ *defined by*

$$\hat{\mu}(f(t)) = \mu(t^{-1}f^{**}(t^{-1})), \tag{1.1}$$

is a r.i. *norm on* $\mathbf{R}$ *whose indices* $(\hat{\beta}, \hat{\alpha})$ *are conjugate to those of* μ, *i.e.*,

$$\alpha + \hat{\beta} = \beta + \hat{\alpha} = 1. \tag{1.2}$$

Furthermore, the Fourier transform is a bounded linear operator from $L^\mu(\mathbf{R})$ *into* $L^{\hat{\mu}}(\mathbf{R})$.

It is a simple matter to check that our result contains Theorem 1.3 as a special case.

There are analogs of Theorem 1.4 for the Fourier transform defined on other locally compact abelian groups (cf. Section 3); in particular, for the Fourier transform defined on the circle group $\mathbf{T}$ by

$$c_n = (\mathscr{F}f)(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\, e^{-\mathrm{int}}\, dt, \qquad n = 0, \pm 1, \pm 2, \ldots, f \in L^1(T).$$

The following estimate was established by Hardy and Littlewood [6] for the Fourier coefficients of functions f of class $L(\log^+ L)$, i.e., functions f on $\mathbf{T}$ for which $|f| \log^+ |f|$ is integrable.

THEOREM 1.5 (Hardy–Littlewood). *If* $f \in L(\log^+ L)$, *then* $\sum_{n=1}^{\infty} n^{-1} c_n^* < \infty$.

This result is not a consequence of Theorem 1.4 (or, more precisely, of its analog for the circle) since the indices of the space $L(\log^+ L)$ are both equal to 1. However, using related techniques we can derive an extension for the classes $L(\log^+ L)^q$, $q > 0$. Furthermore, using Theorem 1.4 directly, we can prove analogous results for the spaces $L^p(\log^+ L)^q$ when $1 < p < 2$.

THEOREM 1.6.

(a) *If* $f \in L(\log^+ L)^q$, $q > 0$, *then* $\sum_{n=2}^{\infty} n^{-1} c_n^*(\log n)^{q-1} < \infty$.

(b) *If* $f \in L^p(\log^+ L)^q$, $1 < p < 2$, $q \geq 0$, *then* $\sum_{n=1}^{\infty} n^{p-2} c_n^{*p}(\log n)^q < \infty$.

Note that when $q = 1$, part (a) yields Theorem 1.5, and when $q = 0$, part (b) reduces to the situation in Theorem 1.2.

Our method of proof for these results is interpolation-theoretic. In particular, we use the $(\rho; k)$ interpolation methods introduced in Part I of this paper [3] and the weak-interpolation theory developed in Part II [4]. We begin with a brief summary.

2. THE $(\rho; k)$ INTERPOLATION METHODS

For an arbitrary compatible couple (X_1, X_2) of Banach spaces we denote by X_1^0 the closure of $X_1 \cap X_2$ in X_1. Whenever $f \in X_1^0 + X_2$, Peetre's K-functional

$$K(t; f) = K(t; f; X_1, X_2) = \inf_{f = f_1 + f_2} (\|f_1\|_1 + t\|f_2\|_2), \tag{2.1}$$

tends to zero with t. It therefore has a representation

$$K(t; f) = \int_0^t k(s; f)\, ds, \qquad 0 < t < \infty, \tag{2.2}$$

where $s \to k(s; f)$ is nonnegative, nonincreasing and right-continuous on $(0, \infty)$ [3, Section 5].

To each r.i. norm ρ on $(0, \infty)$ there corresponds a Banach space $(X_1, X_2)_{\rho;k}$ consisting of those elements f in $X_1^0 + X_2$ for which the norm $\|f\|_{\rho;k} = \rho(k(t; f))$ is finite. The spaces so constructed are intermediate between X_1 and X_2, i.e.,

$$X_1 \cap X_2 \subseteq (X_1, X_2)_{\rho;k} \subseteq X_1 + X_2,$$

and share the following interpolation property [3, Theorem 5.3].

THEOREM 2.1. *Let* (X_1, X_2) *and* (Y_1, Y_2) *be compatible couples and let* $T: X_1 + X_2 \to Y_1 + Y_2$ *be a linear operator which is bounded from* X_i *into* Y_i, $i = 1, 2$. *Then for any* r.i. *norm* ρ *on* $(0, \infty)$, T *is bounded from* $(X_1, X_2)_{\rho;k}$ *into* $(Y_1, Y_2)_{\rho;k}$.

In a similar manner, the space $(X_1, X_2)_{\rho;K}$ consists of those f in $X_1 + X_2$ for which the norm $\|f\|_{\rho;K} = \rho(t^{-1}K(t; f))$ is finite. It is an intermediate space of X_1 and X_2 if the upper index of ρ satisfies $\alpha_\rho < 1$ [3, Corollary 8.8], and there is the following "equivalence theorem" for the spaces generated by the $(\rho; k)$ and $(\rho; K)$ methods [3, Theorem 9.3].

LEMMA 2.2. *If the indices of* ρ *satisfy* $0 < \beta \leqslant \alpha < 1$, *then* $(X_1, X_2)_{\rho;k} = (X_1, X_2)_{\rho;K}$, *with equivalent norms.*

3. THE HAUSDORFF–YOUNG THEOREM

From the discussion in Section 1 we have that the Fourier transform is a bounded linear operator:

$$\mathscr{F}: L^1(\mathbf{R}) \to L^\infty(\mathbf{R}); \qquad \mathscr{F}: L^2(\mathbf{R}) \to L^2(\mathbf{R}).$$

The action of $\mathscr{F}$ on the intermediate spaces $(L^1(\mathbf{R}), L^2(\mathbf{R}))_{\rho;k}$ is then described by Theorem 2.1 as follows.

THEOREM 3.1. *Let* ρ *be any* r.i. *norm on* $(0, \infty)$. *Then the Fourier transform* $\mathscr{F}$ *is a bounded linear operator*:

$$\mathscr{F}: (L^1(\mathbf{R}), L^2(\mathbf{R}))_{\rho;k} \to (L^\infty(\mathbf{R}), L^2(\mathbf{R}))_{\rho;k}. \tag{3.1}$$

Our plan is to characterize the spaces appearing in (3.1) by means of the weak-interpolation theory developed in [4].

LEMMA 3.2 [4, Theorem 3.5; 3, Theorem 10.2]. *Let* $L^\mu(\mathbf{R})$ *be a* r.i. *space whose indices satisfy* $\frac{1}{2} < \beta_\mu \leqslant \alpha_\mu < 1$, *and define* ρ *on* $(0, \infty)$ *by*

$$\rho(f(t)) = \mu(t^{-1/2}f^{**}(t^{1/2})). \tag{3.2}$$

Then ρ *is a* r.i. *norm on* $(0, \infty)$, *with indices given by*

$$\beta_\rho = 2\beta_\mu - 1; \qquad \alpha_\rho = 2\alpha_\mu - 1, \tag{3.3}$$

such that $L^\mu(\mathbf{R}) = (L^1(\mathbf{R}), L^2(\mathbf{R}))_{\rho;k}$, *with equivalent norms.*

LEMMA 3.3 [4, Theorem 3.5; 3, Theorem 10.1]. *Let* σ *be a* r.i. *norm on* $(0, \infty)$ *whose indices satisfy* $0 < \beta_\sigma \leqslant \alpha_\sigma < 1$, *and define* ν *on* $\mathbf{R}$ *by*

$$\nu(f(t)) = \nu(f^*(t)) = \sigma(f^*(t^2)). \tag{3.4}$$

Then ν *is a* r.i. *norm on* $\mathbf{R}$, *with indices given by*

$$\beta_\nu = \tfrac{1}{2}\beta_\sigma; \qquad \alpha_\nu = \tfrac{1}{2}\alpha_\sigma, \tag{3.5}$$

such that $(L^2(\mathbf{R}), L^\infty(\mathbf{R}))_{\sigma;k} = L^\nu(\mathbf{R})$, *with equivalent norms.*

Proof of Theorem 1.4. Let $L^\mu(\mathbf{R})$ be a r.i. space whose indices satisfy

$$\tfrac{1}{2} < \beta_\mu \leqslant \alpha_\mu < 1. \tag{3.6}$$

Then by Lemma 3.2, there is a r.i. norm ρ on $(0, \infty)$ (defined by (3.2)), with indices given by (3.3), such that $L^\mu(\mathbf{R}) = (L^1(\mathbf{R}), L^2(\mathbf{R}))_{\rho;k}$. By Theorem 3.1 we see that $\mathscr{F}$ is a bounded linear operator:

$$\mathscr{F}\colon L^\mu(\mathbf{R}) \to (L^\infty(\mathbf{R}), L^2(\mathbf{R}))_{\rho;k}, \tag{3.7}$$

so if we denote this last space by $L^{\hat{\mu}}$, we have that $\mathscr{F}$ is bounded from L^μ into $L^{\hat{\mu}}$ as desired. It remains therefore to show that $\hat{\mu}$ is a r.i. norm given by (1.1) and whose indices satisfy (1.2).

The first step is to invert the order of the spaces L^∞ and L^2 in (3.7). An inspection of (3.3) and (3.6) reveals that the indices of ρ lie strictly between 0 and 1. Hence, by Lemma 2.2, the $(\rho; k)$ and $(\rho; K)$ methods are equivalent, i.e., $L^{\hat{\mu}} = (L^\infty(\mathbf{R}), L^2(\mathbf{R}))_{\rho;K}$.

It follows directly from (2.1) that

$$K(t; f; L^\infty(\mathbf{R}), L^2(\mathbf{R})) = tK(t^{-1}; f; L^2(\mathbf{R}), L^\infty(\mathbf{R})), \qquad 0 < t < \infty.$$

Thus, if σ is defined over $(0, \infty)$ by

$$\sigma(f(t)) \equiv \sigma(f^*(t)) = \rho(t^{-1} f^{**}(t^{-1})), \tag{3.8}$$

a simple computation using (2.2) shows that

$$L^{\hat{\mu}}(\mathbf{R}) = (L^\infty(\mathbf{R}), L^2(\mathbf{R}))_{\rho;K} = (L^2(\mathbf{R}), L^\infty(\mathbf{R}))_{\sigma;k}. \tag{3.9}$$

Furthermore, it can be checked directly from (3.8) that σ is a r.i. norm on $(0, \infty)$ whose indices are conjugate to those of ρ. Hence, by (3.3)

$$\beta_\sigma = 1 - \alpha_\rho = 2(1 - \alpha_\mu); \qquad \alpha_\sigma = 1 - \beta_\rho = 2(1 - \beta_\mu). \tag{3.10}$$

We next use Lemma 3.3 to find a r.i. norm ν (defined by (3.4)), with indices given by (3.5), such that (using (3.9))

$$L^{\hat{\mu}}(\mathbf{R}) = (L^2(\mathbf{R}), L^\infty(\mathbf{R}))_{\sigma;k} = L^\nu(\mathbf{R}).$$

Thus $\hat{\mu}$ is equivalent to ν and so has the same indices. But by (3.5) and (3.10), the indices of ν are given by

$$\beta_\nu = \tfrac{1}{2}\beta_\sigma = 1 - \alpha_\mu; \qquad \alpha_\nu = \tfrac{1}{2}\alpha_\sigma = 1 - \beta_\mu.$$

Hence, the indices of $\hat{\mu}$ are conjugate to those of μ.

Finally, we note that the indices of all the norms involved are strictly between 0 and 1. Hence, as in the proof of [3, Theorem 10.2], we can replace

f^{**} by f^* in the definitions (3.2) and (3.8) to obtain equivalent quasinorms. But then from (3.2), (3.4) and (3.8) we find that

$$\hat{\mu}(f(t)) \sim \nu(f(t)) \sim \mu(t^{-1}f^*(t^{-1})). \tag{3.11}$$

The functional on the right may fail to satisfy the triangle inequality so we replace f^* by f^{**} to obtain the desired representation (1.1) for $\hat{\mu}$. This completes the proof.

Remarks.

(i) A somewhat weaker version of Theorem 1.4 was obtained previously by the author in [2].

(ii) There are analogs of Theorem 1.4 for the Fourier transform defined on locally compact abelian groups **G** (cf. [8]) whose Haar measure is σ-finite. In this case, the identity (1.1) needs slight but obvious modification (cf. Section 4 where we apply it for the circle group).

4. The Spaces $L^p(\log^+ L)^q$

For spaces with indices equal to 1, the arguments used in the last section fail. However, in certain cases we can appeal directly to Theorem 3.1. To illustrate our technique we consider the r.i. spaces $Z^{pq} = L^p(\log^+ L)^q$, $1 \leqslant p < 2$, $0 \leqslant q < \infty$. For these values of p and q, Z^{pq} is an Orlicz space whose indices are both equal to p^{-1}. The next lemma shows that Z^{pq} is also a Lorentz space. The proof is elementary and can be found in [1].

Lemma 4.1. *Let f be an integrable function on the circle and suppose $1 \leqslant p < 2$, $q \geqslant 0$. Then the following assertions are equivalent.*

(i) $f \in Z^{pq}$;

(ii) $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p (\log^+ |f(x)|)^q \, dx < \infty$;

(iii) $\displaystyle \int_0^1 f^*(t)^p (\log^+ f^*(t))^q \, dt < \infty$;

(iv) $\displaystyle \int_0^1 f^*(t)^p (\log 1/t)^q \, dt < \infty$.

If, in addition, we have $p = 1$ and $q > 0$, then each of the above is equivalent to

(v) $\displaystyle \int_0^1 f^{**}(t)(\log 1/t)^{q-1} \, dt < \infty$.

It follows from the lemma that the functional

$$\mu(f) = \left(\int_0^1 f^*(t)^p (\log 1/t)^q \, dt\right)^{1/p}, \qquad f \in Z^{pq}, \tag{4.1}$$

defines an equivalent norm on Z^{pq} under which Z^{pq} becomes a Lorentz space.

We first deal with the case $1 < p < 2$, since then Theorem 1.4 applies directly.

THEOREM 4.2. *Suppose* $1 < p < 2$ *and* $q \geqslant 0$. *If* $f \in Z^{pq}$, *then*

$$\sum_{n=1}^{\infty} n^{p-2} c_n^{*p} (\log n)^q < \infty, \tag{4.2}$$

where $\{c_n^*\}$ *is the decreasing rearrangement of the sequence* $\{c_n\}$ *of Fourier coefficients of* f.

Proof. The correct interpretation of (1.1) is as follows. If $f \in Z^{pq}$ and $\{c_n\}$ is its sequence of Fourier coefficients we set

$$g(t) = c_n^*, \qquad n-1 \leqslant t < n, \qquad n = 1, 2, \ldots. \tag{4.3}$$

Then by (4.1)

$$\hat{\mu}(\{c_n\}) \equiv \hat{\mu}(\{c_n^*\}) = \mu(t^{-1}g^{**}(t^{-1})) = \left\{\int_0^1 [t^{-1}g^{**}(t^{-1})]^p (\log 1/t)^q \, dt\right\}^{1/p}.$$

But by (4.3)

$$\begin{aligned}\int_0^1 [t^{-1}g^{**}(t^{-1})]^p (\log 1/t)^q \, dt &= \int_1^\infty [tg^{**}(t)]^p (\log t)^q \, dt/t^2 \\ &\geqslant \int_1^\infty t^{p-2} g^*(t)^p (\log t)^q \, dt \\ &\geqslant \sum_{n=2}^{\infty} c_n^{*p} n^{p-2} (\log(n-1))^q.\end{aligned}$$

Hence, since $\mathscr{F}$ is a bounded operator from L^μ into $L^{\hat{\mu}}$ we have

$$\sum_{n=2}^{\infty} c_n^{*p} n^{p-2} (\log(n-1))^q \leqslant \hat{\mu}(\{c_n\})^p \leqslant c\mu(f)^p < \infty,$$

from which the desired conclusion (4.2) follows.

THEOREM 4.3. *If* $f \in Z^{1q} = L(\log^+ L)^q$, $q > 0$, *then*

$$\sum_{n=2}^{\infty} n^{-1} c_n^* (\log n)^{q-1} < \infty.$$

Proof. Fix $q > 0$ and define the r.i. norm ρ on $(0, \infty)$ by

$$\rho(\varphi) = \int_0^1 (\log 1/t)^{q-1} \varphi^{**}(t)\, dt. \tag{4.4}$$

The Fourier transform maps L^2 isometrically onto l^2 but it will be more convenient to work with the weaker hypothesis $\mathscr{F}: L^{21} \to l^{2\infty}$. In this case the interpolation theorem (Theorem 2.1) shows that $\mathscr{F}$ is a bounded linear operator:

$$\mathscr{F}: (L^1, L^{21})_{\rho;k} \to (l^\infty, l^{2\infty})_{\rho;k}. \tag{4.5}$$

We first show that $(L^1, L^{21})_{\rho;k}$ is precisely the space Z^{1q} up to equivalence of norms. For this we use Holmstedt's characterization [7] of the K-functional $K(t; f; L^1, L^{21})$:

$$K(t; f) \sim t \int_{t^2}^{\infty} s^{1/2} f^{**}(s)\, ds/s, \qquad 0 < t < \infty.$$

Combining (2.2), (4.4) and the last estimate, we have for any $f \in (L^1, L^{21})_{\rho;k}$,

$$\|f\|_{\rho;k} = \rho(k(t;f)) = \int_0^1 (\log 1/t)^{q-1}\, t^{-1} K(t;f)\, dt$$
$$\sim \int_0^1 (\log 1/t)^{q-1}\, dt \int_{t^2}^{\infty} s^{1/2} f^{**}(s)\, ds/s.$$

Interchanging the order of integration and remembering that f^* is supported on $[0, 1]$ we find that

$$\|f\|_{\rho;k} \sim \int_0^1 (\log 1/s)^{q-1} f^{**}(s)\, ds,$$

so Lemma 4.1 yields the equivalence $Z^{1q} \sim (L^1, L^{21})_{\rho;k}$.

Next, we turn to the space $(l^\infty, l^{2\infty})_{\rho;k}$. In this case, Holmstedt's theorem [7] gives

$$t^{-1} K(t; g; l^\infty, l^{2\infty}) \sim \sup_{s \geq t^2} s^{-1/2} K(s; g; l^\infty, l^1)$$
$$= \sup_{s \geq t^2} s^{-1/2} g^{**}(s^{-1}) = \sup_{s \leq t^{-2}} s^{1/2} g^{**}(s).$$

Hence, the norm of any function $g \in (l^\infty, l^{2\infty})_{\rho;k}$ satisfies

$$\|g\|_{\rho;k} = \rho(k(t; g)) \sim \int_0^1 (\log 1/t)^{q-1} \sup_{s \leq t^{-2}} [s^{1/2} g^{**}(s)]\, dt$$
$$= \int_1^{\infty} t^{-2} (\log t)^{q-1} \sup_{s \leq t^2} [s^{1/2} g^{**}(s)]\, dt.$$

The last integral has lower bound

$$\int_1^\infty t^{-2} (\log t)^{q-1}\, tg^{**}(t^2)\, dt \geqslant \int_1^\infty t^{-1} (\log t)^{q-1}\, g^*(t^2)\, dt$$
$$= 2^{-q} \int_1^\infty (\log t)^{q-1}\, g^*(t)\, dt/t,$$

from which it follows easily that

$$\sum_{n=1}^{\infty} n^{-1} (\log n)^{q-1}\, g^*(n) < \infty, \qquad \text{if} \quad g \in (l^\infty, l^{2\infty})_{\rho;k}\,. \tag{4.6}$$

Hence, if $f \in Z^{1q} = (L^1, L^{21})_{\rho;k}$, then by (4.5) its sequence $\{c_n\}$ of Fourier coefficients lies in $(l^\infty, l^{2\infty})_{\rho;k}$ so from (4.6) we deduce that $\sum_{n=1}^{\infty} n^{-1} c_n^* (\log n)^{q-1} < \infty$. This completes the proof.

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